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Weyl groups of fine gradings on matrix algebras, octonions and the Albert algebra

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ABSTRACT

Given a grading $\Gamma: \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ on a nonassociative algebra \mathcal{A} by an abelian group G , we have two subgroups of $\text{Aut}(\mathcal{A})$: the automorphisms that stabilize each component \mathcal{A}_g (as a subspace) and the automorphisms that permute the components. By the Weyl group of Γ we mean the quotient of the latter subgroup by the former. In the case of a Cartan decomposition of a semisimple complex Lie algebra, this is the automorphism group of the root system, i.e., the so-called extended Weyl group. A grading is called fine if it cannot be refined. We compute the Weyl groups of all fine gradings on matrix algebras, octonions and the Albert algebra over an algebraically closed field (of characteristic different from 2 in the case of the Albert algebra).

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1. Introduction

Let \mathcal{A} be an algebra (not necessarily associative) over a field \mathbb{F} and let G be a group. We will usually use multiplicative notation for G , but switch to additive notation when working with groups such as \mathbb{Z} and $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$.

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Definition 1.1. A G -grading on \mathcal{A} is a vector space decomposition

$$\Gamma: \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$$

such that

$$\mathcal{A}_g \mathcal{A}_h \subset \mathcal{A}_{gh} \quad \text{for all } g, h \in G.$$

If such a decomposition is fixed, we will refer to \mathcal{A} as a G -graded algebra. The nonzero elements $a \in \mathcal{A}_g$ are said to be *homogeneous of degree g* ; we will write $\deg a = g$. The *support* of Γ is the set $\text{Supp } \Gamma := \{g \in G \mid \mathcal{A}_g \neq 0\}$.

There are two natural ways to define equivalence relation on group graded algebras. We will use the term “isomorphism” for the case when the grading group is a part of definition and “equivalence” for the case when the grading group plays a secondary role. Let

$$\Gamma: \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g \quad \text{and} \quad \Gamma': \mathcal{B} = \bigoplus_{h \in H} \mathcal{B}_h$$

be two gradings on algebras, with supports S and T , respectively.

Definition 1.2. We say that Γ and Γ' are *equivalent* if there exists an isomorphism of algebras $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ and a bijection $\alpha: S \rightarrow T$ such that $\varphi(\mathcal{A}_s) = \mathcal{B}_{\alpha(s)}$ for all $s \in S$. Any such φ will be called an *equivalence* of Γ and Γ' . If the gradings on \mathcal{A} and \mathcal{B} are clear from the context, then we will say that \mathcal{A} and \mathcal{B} are *equivalent*.

The algebras graded by a fixed group G form a category where the morphisms are the *homomorphisms of G -graded algebras*, i.e., homomorphisms of algebras $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ such that $\varphi(\mathcal{A}_g) \subset \mathcal{B}_g$ for all $g \in G$.

Definition 1.3. In the case $G = H$, we say that Γ and Γ' are *isomorphic* if \mathcal{A} and \mathcal{B} are isomorphic as G -graded algebras, i.e., there exists an isomorphism of algebras $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ such that $\varphi(\mathcal{A}_g) = \mathcal{B}_g$ for all $g \in G$.

Replacing G with a smaller group, we can assume that $\text{Supp } \Gamma$ generates G . Even with this assumption, there will be, in general, many other groups H such that the vector space decomposition Γ can be realized as an H -grading. It turns out [PZ89] that there is one distinguished group among them.

Definition 1.4. Suppose that Γ admits a realization as a G_0 -grading for some group G_0 . We will say that G_0 is a *universal group* of Γ if for any other realization of Γ as a G -grading, there exists a unique homomorphism $G_0 \rightarrow G$ that restricts to identity on $\text{Supp } \Gamma$. We define the *universal abelian group* in the same manner.

Note that, by definition, G_0 is a group with a distinguished generating set, $\text{Supp } \Gamma$. A standard argument shows that, if a universal (abelian) group exists, it is unique up to an isomorphism over $\text{Supp } \Gamma$. We will denote it by $U(\Gamma)$. It turns out that $U(\Gamma)$ exists and depends only on the equivalence class of Γ . Details may be found in [Koc09].

Following [PZ89], we can associate three subgroups of $\text{Aut}(\mathcal{A})$ to a grading Γ on an algebra \mathcal{A} .

Definition 1.5. The *automorphism group* of Γ , denoted by $\text{Aut}(\Gamma)$, consists of all automorphisms of \mathcal{A} that permute the components of Γ . Each $\varphi \in \text{Aut}(\Gamma)$ determines a self-bijection $\alpha = \alpha(\varphi)$ of the support S such that $\varphi(\mathcal{A}_s) = \mathcal{A}_{\alpha(s)}$ for all $s \in S$. The *stabilizer* of Γ , denoted by $\text{Stab}(\Gamma)$, is the kernel of the homomorphism $\text{Aut}(\Gamma) \rightarrow \text{Sym}(S)$ given by $\varphi \mapsto \alpha(\varphi)$. Finally, the *diagonal group* of Γ , denoted by $\text{Diag}(\Gamma)$, is the subgroup of the stabilizer consisting of all automorphisms φ such that the restriction of φ to any homogeneous component of Γ is the multiplication by a (nonzero) scalar.

Thus $\text{Aut}(\Gamma)$ is the group of self-equivalences of the graded algebra \mathcal{A} and $\text{Stab}(\Gamma)$ is the group of automorphisms of the graded algebra \mathcal{A} . $\text{Diag}(\Gamma)$ is isomorphic to the group of characters of $U(\Gamma)$. If $\dim \mathcal{A} < \infty$, then $\text{Diag}(\Gamma)$ is a diagonalizable algebraic group (quasitorus). If, in addition, \mathbb{F} is an algebraically closed field of characteristic 0 and G is abelian, then Γ is the eigenspace decomposition of \mathcal{A} relative to $\text{Diag}(\Gamma)$ – see e.g. [Koc09], the group $\text{Stab}(\Gamma)$ is the centralizer of $\text{Diag}(\Gamma)$, and $\text{Aut}(\Gamma)$ is its normalizer.

Definition 1.6. The quotient group $\text{Aut}(\Gamma)/\text{Stab}(\Gamma)$, which is a subgroup of $\text{Sym}(S)$, will be called the *Weyl group* of Γ and denoted by $W(\Gamma)$.

We use the term “Weyl group”, because if Γ is the Cartan grading on a semisimple complex Lie algebra \mathfrak{g} , then $W(\Gamma)$ is isomorphic to the so-called extended Weyl group of \mathfrak{g} , i.e., the automorphism group of the root system of \mathfrak{g} .

It follows from the universal property of $U(\Gamma)$ that any $\varphi \in \text{Aut}(\Gamma)$ gives rise to a unique automorphism $u(\varphi)$ of $U(\Gamma)$ such that the following diagram commutes:

$$\begin{array}{ccc} \text{Supp } \Gamma & \longrightarrow & U(\Gamma) \\ \alpha(\varphi) \downarrow & & \downarrow u(\varphi) \\ \text{Supp } \Gamma & \longrightarrow & U(\Gamma) \end{array}$$

where the horizontal arrows are the canonical imbeddings. This gives an action of $\text{Aut}(\Gamma)$ by automorphisms of the group $U(\Gamma)$. The kernel of this action is $\text{Stab}(\Gamma)$, so we may regard $W(\Gamma) = \text{Aut}(\Gamma)/\text{Stab}(\Gamma)$ as a subgroup of $\text{Aut}(U(\Gamma))$.

A grading Γ is said to be *fine* if it cannot be refined in the class of (abelian) group gradings. Any G -grading on a finite-dimensional algebra \mathcal{A} is induced from some fine grading Γ by a homomorphism $\alpha: U(\Gamma) \rightarrow G$ as follows: $\mathcal{A}_g = \bigoplus_{u \in \alpha^{-1}(g)} \mathcal{A}_u$ for all $g \in G$.

From now on, we assume that all grading groups are *abelian* and the ground field \mathbb{F} is *algebraically closed*. A description of fine gradings on the matrix algebras $M_n(\mathbb{F})$ was obtained in [HPP98,BSZ01] for characteristic 0 and extended to characteristic p in [BZ03]. All fine gradings on the octonion algebra \mathcal{C} were described in [Eld98]. All fine gradings on the Albert algebra \mathcal{A} in characteristic 0 were classified in [DM09]; the same classification was shown to be valid in characteristic $p \neq 2$ in [EK12].

The Weyl groups were computed for some special cases of fine gradings on $M_n(\mathbb{C})$ in [HPPT02, PST06,Han10]. Here we compute them for all fine gradings on $M_n(\mathbb{F})$ over any algebraically closed field \mathbb{F} – see Section 2. In Section 3, we compute the Weyl groups of fine gradings on the algebra of octonions. In Section 4, we compute them for the Albert algebra assuming that the characteristic is not 2.

2. Matrix algebras

The goal of this section is to compute the Weyl groups of all fine gradings on the matrix algebra $M_n(\mathbb{F})$. The proof will use graded modules over $M_n(\mathbb{F})$, so we will first state the classification of fine gradings in that language. All algebras in this section will be assumed *associative*. All algebras and modules will be assumed *finite-dimensional*.

2.1. Gradings on matrix algebras

Let G be a group. A vector space V is G -graded if it is equipped with a decomposition $V = \bigoplus_{g \in G} V_g$. A *graded subspace* $W \subset V$ is a subspace satisfying $W = \bigoplus_{g \in G} (V_g \cap W)$, so W inherits a G -grading from V . We extend Definitions 1.2 and 1.3 to graded vector spaces. For $g \in G$, we define the *shift* $V^{[g]}$ of a G -graded vector space V by setting $V_{hg}^{[g]} := V_h$, $h \in G$.

Let \mathcal{R} be a G -graded algebra. A *graded left \mathcal{R} -module* is a left \mathcal{R} -module V that is also a G -graded vector space such that $\mathcal{R}_g V_h \subset V_{gh}$ for all $g, h \in G$. A *graded right \mathcal{R} -module* is defined similarly. A *homomorphism of graded \mathcal{R} -modules* $f: V \rightarrow W$ is a homomorphism of \mathcal{R} -modules such that $f(V_g) \subset W_g$ for all $g \in G$.

We will follow the convention of writing homomorphisms of left modules on the right and homomorphisms of right modules on the left. Let V and W be graded left \mathcal{R} -modules. Regarding V and W as G -graded vector spaces, we have the graded space $\text{Hom}(V, W) = \bigoplus_{g \in G} \text{Hom}_g(V, W)$ where

$$\text{Hom}_g(V, W) := \{f: V \rightarrow W \mid (V_h)f \subset W_{hg} \text{ for all } h \in G\}.$$

The space $\text{Hom}_{\mathcal{R}}(V, W)$ is a graded subspace in $\text{Hom}(V, W)$. When $W = V$, we obtain a G -graded algebra $\text{End}_{\mathcal{R}}(V) := \text{Hom}_{\mathcal{R}}(V, V)$.

A G -graded algebra \mathcal{D} is said to be a *graded division algebra* if it is unital and every nonzero homogeneous element has an inverse. Let $T \subset G$ be the support of \mathcal{D} . Then T is a subgroup of G . Any graded \mathcal{D} -module V is free and can be decomposed canonically into the direct sum of (nonzero) isotypical components:

$$V = V_1 \oplus \cdots \oplus V_s$$

where V_i is the sum of all graded submodules that are isomorphic to some fixed $\mathcal{D}^{[g_i]}$, $g_i \in G$. The elements g_1, \dots, g_s are not uniquely determined, but their cosets $g_1 T, \dots, g_s T$ are determined up to permutation. Write

$$\gamma = (g_1, \dots, g_s) \quad \text{where } g_i^{-1} g_j \notin T \text{ for } i \neq j. \quad (1)$$

If $\{v_1, \dots, v_n\}$ is a homogeneous \mathcal{D} -basis in V , then, for each i , the subset

$$\{v_j \mid \deg v_j \in g_i T\}$$

is a \mathcal{D} -basis for V_i . Let $k_i = \dim_{\mathcal{D}} V_i$ and write

$$\kappa = (k_1, \dots, k_s). \quad (2)$$

Conversely, for a given pair (κ, γ) , let $V(G, \mathcal{D}, \kappa, \gamma)$ be the right \mathcal{D} -module that has a homogeneous \mathcal{D} -basis consisting of k_i elements of degree g_i , $i = 1, \dots, s$. Denote the G -graded algebra $\text{End}_{\mathcal{D}}(V)$ by $\mathcal{M}(G, \mathcal{D}, \kappa, \gamma)$.

We will say that a G -grading on $M_n(\mathbb{F})$ is a *division grading* if $M_n(\mathbb{F})$ is a graded division algebra. Such gradings can be constructed as follows. Suppose we have a finite subgroup $T \subset G$ such that there exists a *nondegenerate alternating bicharacter* $\beta: T \times T \rightarrow \mathbb{F}^\times$, i.e., a function that is multiplicative in each variable and satisfies the following two properties: $\beta(t, t) = 1$ for all $t \in T$ (alternating) and $\beta(u, T) = 1$ implies $u = e$ (nondegenerate). Then T admits a “symplectic basis”, i.e., there exists a decomposition of T into the direct product of cyclic subgroups:

$$T = (H'_1 \times H''_1) \times \cdots \times (H'_r \times H''_r) \quad (3)$$

such that $H'_i \times H''_i$ and $H'_j \times H''_j$ are β -orthogonal for $i \neq j$, and H'_i and H''_i are in duality by β . Denote by ℓ_i the order of H'_i and H''_i . (We may assume without loss of generality that ℓ_i are prime powers.) If we pick generators a_i and b_i for H'_i and H''_i , respectively, then $\varepsilon_i := \beta(a_i, b_i)$ is a primitive ℓ_i -th root of unity, and all other values of β on the elements $a_1, b_1, \dots, a_r, b_r$ are 1. Define the following elements of the algebra $M_{\ell_1}(\mathbb{F}) \otimes \dots \otimes M_{\ell_r}(\mathbb{F})$:

$$X_{a_i} = I \otimes \dots \otimes I \otimes X_i \otimes I \otimes \dots \otimes I \quad \text{and} \quad X_{b_i} = I \otimes \dots \otimes I \otimes Y_i \otimes I \otimes \dots \otimes I,$$

where

$$X_i = \begin{bmatrix} \varepsilon_i^{n-1} & 0 & 0 & \dots & 0 & 0 \\ 0 & \varepsilon_i^{n-2} & 0 & \dots & 0 & 0 \\ \dots & & & & & \\ 0 & 0 & 0 & \dots & \varepsilon_i & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \quad \text{and} \quad Y_i = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & & & & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}$$

are the generalized Pauli matrices in the i -th factor, $M_{\ell_i}(\mathbb{F})$. Finally, set

$$X_{(a_1^{i_1}, b_1^{j_1}, \dots, a_r^{i_r}, b_r^{j_r})} = X_{a_1}^{i_1} X_{b_1}^{j_1} \dots X_{a_r}^{i_r} X_{b_r}^{j_r}.$$

Identify $M_{\ell_1}(\mathbb{F}) \otimes \dots \otimes M_{\ell_r}(\mathbb{F})$ with $M_\ell(\mathbb{F})$, $\ell = \ell_1 \dots \ell_r$, via Kronecker product. Then

$$M_\ell(\mathbb{F}) = \bigoplus_{t \in T} \mathbb{F} X_t \quad (4)$$

is a division grading with support T . Note that by construction we have

$$X_u X_v = \beta(u, v) X_v X_u \quad \text{for all } u, v \in T. \quad (5)$$

Theorem 2.1. (See [BSZ01, BZ03].) *Let T be a finite abelian group and let \mathbb{F} be an algebraically closed field. There exists a division grading on the matrix algebra $M_\ell(\mathbb{F})$ with support T if and only if $\text{char } \mathbb{F}$ does not divide ℓ and $T \cong \mathbb{Z}_{\ell_1}^2 \times \dots \times \mathbb{Z}_{\ell_r}^2$ where $\ell_1 \dots \ell_r = \ell$. All such gradings are equivalent to the grading given by (4). \square*

Theorem 2.2. (See [BSZ01, BZ03].) *Let G be an abelian group and let \mathcal{R} be a matrix algebra with a G -grading. Then \mathcal{R} is isomorphic to some $\mathcal{M}(G, \mathcal{D}, \kappa, \gamma)$ where \mathcal{D} is a matrix algebra with division grading, and κ and γ are as in (1) and (2). \square*

The following notion and result will be crucial to our computation of Weyl groups.

Definition 2.3. Let G and H be groups. Let \mathcal{D} be a G -graded algebra and \mathcal{D}' an H -graded algebra. Let V be a graded right \mathcal{D} -module and V' a graded right \mathcal{D}' -module. An equivalence from (\mathcal{D}, V) to (\mathcal{D}', V') is a pair (ψ_0, ψ_1) such that $\psi_0: \mathcal{D} \rightarrow \mathcal{D}'$ is an equivalence of graded algebras and $\psi_1: V \rightarrow V'$ is an equivalence of graded vector spaces, and $\psi_1(vd) = \psi_1(v)\psi_0(d)$ for all $v \in V$ and $d \in \mathcal{D}$.

Proposition 2.4. *Let G and H be groups. Let \mathcal{D} be a G -graded algebra and \mathcal{D}' an H -graded algebra. Suppose that \mathcal{D} and \mathcal{D}' are graded division algebras. Let V be a graded right \mathcal{D} -module and V' a graded right \mathcal{D}' -module. Let $\mathcal{R} = \text{End}_{\mathcal{D}}(V)$ and $\mathcal{R}' = \text{End}_{\mathcal{D}'}(V')$. If $\psi: \mathcal{R} \rightarrow \mathcal{R}'$ is an equivalence of graded algebras, then there exists an equivalence (ψ_0, ψ_1) from (\mathcal{D}, V) to (\mathcal{D}', V') such that $\psi_1(rv) = \psi(r)\psi_1(v)$ for all $r \in \mathcal{R}$ and $v \in V$.*

Proof. Let $I \subset \mathcal{R}$ be a minimal graded left ideal. A standard argument shows that I is generated by a homogeneous idempotent. Indeed, we have $I^2 \neq 0$, because otherwise $I\mathcal{R}$ would be a graded two-sided ideal with the property $(I\mathcal{R})^2 = 0$, which is impossible because \mathcal{R} is graded simple. Pick a homogeneous $x \in I$ such that $Ix \neq 0$. By the minimality of I , we have $Ix = I$ and $\text{ann}_I(x) = 0$, where $\text{ann}_I(x) := \{r \in I \mid rx = 0\}$. Hence there exists $e \in I$ such that $ex = x$. Replacing e by its identity component, we may assume that e is homogeneous. Since $e^2 - e \in \text{ann}_I(x)$, we conclude that $e^2 = e$. Since $\mathcal{R}e \neq 0$, we have $\mathcal{R}e = I$ by minimality.

Since V is graded simple as a left \mathcal{R} -module and IV is a graded submodule of V , we have either $IV = 0$ or $IV = V$. But the action of \mathcal{R} on V is faithful, so $IV = V$. Pick a homogeneous $v \in V$ such that $Iv \neq 0$ and let $g = \deg v$. Then the map $I \rightarrow V$ given by $r \mapsto rv$ is a homomorphism of \mathcal{R} -modules. By graded simplicity of I and V , this map is an isomorphism. It sends I_a to V_{ag} , $a \in G$, so it is an isomorphism of graded \mathcal{R} -modules when regarded as a map $I^{[g]} \rightarrow V$. Now, $\text{End}_{\mathcal{R}}(I)$ can be identified with $e\mathcal{R}e$ (as a graded algebra) in the usual way. Indeed, the right multiplication by a homogeneous $x \in e\mathcal{R}e$ gives an endomorphism of I , which has the same degree as x . Conversely, any endomorphism of I is easily seen to coincide with the right multiplication by the image of e . It follows that we have an isomorphism of graded algebras $e\mathcal{R}e \rightarrow \text{End}_{\mathcal{R}}(V^{[g^{-1}]})$ sending $x \in e\mathcal{R}e$ to the endomorphism $rv \mapsto rxv$, $r \in I$. Since $\mathcal{D} = \text{End}_{\mathcal{R}}(V)$, we have obtained an equivalence $(e\mathcal{R}e, I) \rightarrow (\mathcal{D}, V)$, where $I \rightarrow V$ is an isomorphism of \mathcal{R} -modules.

Since $I' = \psi(I)$ is a minimal graded left ideal of \mathcal{R}' , which is generated by the homogeneous idempotent $e' = \psi(e)$, we also have an equivalence $(e'\mathcal{R}'e', I') \rightarrow (\mathcal{D}', V')$. Finally, restricting ψ yields equivalences $\tilde{\psi}_0: e\mathcal{R}e \rightarrow e'\mathcal{R}'e'$ and $\tilde{\psi}_1: I \rightarrow I'$ such that $\tilde{\psi}_1(rx) = \psi(r)\tilde{\psi}_1(x)$ for $r \in \mathcal{R}$ and $x \in I$. The compositions $\tilde{\psi}_0: \mathcal{D} \rightarrow e\mathcal{R}e \xrightarrow{\tilde{\psi}_0} e'\mathcal{R}'e' \rightarrow \mathcal{D}'$ and $\tilde{\psi}_1: V \rightarrow I \xrightarrow{\tilde{\psi}_1} I' \rightarrow V'$ have the desired properties. \square

The map ψ corresponding to $(\tilde{\psi}_0, \tilde{\psi}_1)$ in Proposition 2.4 can be expressed in the language of matrices as follows. Let $\{v_1, \dots, v_k\}$ be a homogeneous \mathcal{D} -basis in V . Then any $r \in \mathcal{R}$ is represented by a matrix $X = (x_{ij}) \in M_k(\mathcal{D})$ relative to this \mathcal{D} -basis. Also, $\{\tilde{\psi}_1(v_1), \dots, \tilde{\psi}_1(v_k)\}$ is a homogeneous \mathcal{D}' -basis in V' , and ψ is given by

$$M_k(\mathcal{D}) \rightarrow M_k(\mathcal{D}'): (x_{ij}) \mapsto (\psi_0(x_{ij})). \quad (6)$$

From Theorems 2.1 and 2.2, it is easy to obtain all fine gradings on $M_n(\mathbb{F})$. They were described in [HPP98] over the field of complex numbers.

Definition 2.5. Let $\mathcal{D} = M_\ell(\mathbb{F})$ with a division grading (4). Let $k \geq 1$ be an integer. Let \tilde{G} be the direct product of T and the free abelian group generated by the symbols $\tilde{g}_1, \dots, \tilde{g}_k$. Let $\mathcal{M}(\mathcal{D}, k) := \mathcal{M}(\tilde{G}, \mathcal{D}, \kappa, \tilde{\gamma})$ where $\kappa = (1, \dots, 1)$ and $\tilde{\gamma} = (\tilde{g}_1, \dots, \tilde{g}_k)$. Then $\mathcal{M}(\mathcal{D}, k)$ is isomorphic to the matrix algebra $M_n(\mathbb{F})$ where $n = k\ell$. We will denote the \tilde{G} -grading on $M_n(\mathbb{F})$ arising from this isomorphism by $\Gamma_{\mathcal{M}}(\mathcal{D}, k)$ or, abusing notation, by $\Gamma_{\mathcal{M}}(T, k)$, since the equivalence class of \mathcal{D} is uniquely determined by T .

$\Gamma_{\mathcal{M}}(\mathcal{D}, k)$ can be described explicitly as follows. Pick a \mathcal{D} -basis $\{v_1, \dots, v_k\}$ in V with $\deg(v_i) = \tilde{g}_i$, $i = 1, \dots, k$. Then any $r \in \mathcal{R}$ is represented by a matrix $X = (x_{ij}) \in M_k(\mathcal{D})$ relative to this \mathcal{D} -basis. Identify $M_k(\mathcal{D})$ with $M_k(\mathbb{F}) \otimes \mathcal{D}$ via Kronecker product. Let E_{ij} be the matrix units in $M_k(\mathbb{F})$. Then $\Gamma_{\mathcal{M}}(\mathcal{D}, k)$ is given by

$$\deg(E_{ij} \otimes d) = \tilde{g}_i(\deg d)\tilde{g}_j^{-1} \quad \text{for all homogeneous } d \in \mathcal{D}, i, j = 1, \dots, k.$$

This grading is fine in the class of abelian group gradings. The support consists of the elements $\tilde{g}_i t \tilde{g}_j^{-1}$, $t \in T$. The subgroup \tilde{G}^0 that they generate is isomorphic to $\mathbb{Z}^{k-1} \times T$. It is the universal group of $\Gamma_{\mathcal{M}}(\mathcal{D}, k)$.

Corollary 2.6. Let Γ be a fine abelian group grading on the matrix algebra $\mathcal{R} = M_n(\mathbb{F})$ over an algebraically closed field \mathbb{F} . Then Γ is equivalent to some $\Gamma_{\mathcal{M}}(T, k)$ where $T \cong \mathbb{Z}_{\ell_1}^2 \times \cdots \times \mathbb{Z}_{\ell_r}^2$ and $k\ell_1 \cdots \ell_r = n$. Two gradings $\Gamma_{\mathcal{M}}(T_1, k_1)$ and $\Gamma_{\mathcal{M}}(T_2, k_2)$ are equivalent if and only if $T_1 \cong T_2$ and $k_1 = k_2$. \square

2.2. The Weyl group of $\Gamma_{\mathcal{M}}(T, k)$

Let $\mathcal{R} = \mathcal{M}(\mathcal{D}, k)$ and let V be the associated graded module, with a \mathcal{D} -basis $\{v_1, \dots, v_k\}$ where $\deg(v_i) = \tilde{g}_i$, $i = 1, \dots, k$. Note that the homogeneous components of V have the form $\mathbb{F}v_id$ where d is a nonzero homogeneous element of \mathcal{D} . Applying Proposition 2.4 to an equivalence $\psi: \mathcal{R} \rightarrow \mathcal{R}$, we see that there exists an equivalence (ψ_0, ψ_1) of (\mathcal{D}, V) to itself such that $\psi_1(rv) = \psi(r)\psi_1(v)$ for all $r \in \mathcal{R}$ and $v \in V$. Write $\psi_1(v_j) = \sum_i v_id_{ij}$ for some $d_{ij} \in \mathcal{D}$ and set $\Psi := (d_{ij})$. Then (6) implies that $\psi: \mathcal{R} \rightarrow \mathcal{R}$ is given by

$$X \mapsto \Psi \psi_0(X) \Psi^{-1} \quad (7)$$

where ψ_0 acts on X entry-wise. Since ψ_1 is an equivalence of the graded space V to itself, it must send v_i to some v_jd where d is a nonzero homogeneous element of \mathcal{D} . Hence there exists a permutation $\pi \in \text{Sym}(k)$ and nonzero homogeneous d_1, \dots, d_k such that $\psi_1(v_i) = v_{\pi(i)}d_i$. In other words, Ψ is the monomial matrix PD where P is the permutation matrix corresponding to π (i.e., the matrix having 1 in the $(i, \pi^{-1}(i))$ -th positions and zeros elsewhere) and D is the diagonal matrix $\text{diag}(d_1, \dots, d_k)$.

We see that everything boils down to the equivalence ψ_0 of \mathcal{D} to itself. The following result describes all such ψ_0 .

Proposition 2.7. Let \mathcal{D} be a matrix algebra with a division grading Γ_0 given by (4). Let T be the support and let β be the nondegenerate alternating bicharacter $T \times T \rightarrow \mathbb{F}^\times$ determined by (5). Then the mapping that sends $t \in T$ to the inner automorphism $X \mapsto X_t X X_t^{-1}$ is an isomorphism between T and $\text{Stab}(\Gamma_0)$. The quotient group $\text{Aut}(\Gamma_0)/\text{Stab}(\Gamma_0)$ is isomorphic to $\text{Aut}(T, \beta)$.

Proof. The first assertion is proved in [Eld10] and [BK10], but we include a proof for completeness. Any $\psi \in \text{Stab} \Gamma_0$ must send X_t to its scalar multiple, so we have a map $\lambda: T \rightarrow \mathbb{F}^\times$ such that $\psi(X_t) = \lambda(t)X_t$ for all $t \in T$. One immediately verifies that λ must be a group homomorphism, i.e., a character of T . It follows from (5) that the inner automorphism $X \mapsto X_t X X_t^{-1}$ corresponds to the character $\lambda(u) = \beta(t, u)$, $u \in T$. Since β is nondegenerate, it establishes an isomorphism between T and the group of characters \hat{T} .

Any $\psi \in \text{Aut} \Gamma_0$ must send X_t to a scalar multiple of some X_u . Hence we have a homomorphism $f: \text{Aut}(\Gamma_0) \rightarrow \text{Sym}(T)$ such that $\psi(\mathbb{F}X_t) = \mathbb{F}X_{f(\psi)(t)}$ for all $t \in T$. Clearly, the kernel of f is $\text{Stab}(\Gamma_0)$. Fix $\psi \in \text{Aut}(\Gamma_0)$ and let $\pi = f(\psi)$. Since $\psi(X_u X_v) \in \mathbb{F}\psi(X_{uv})$ and $\psi(X_u)\psi(X_v) \in \mathbb{F}X_{\pi(u)\pi(v)}$, we conclude that $\pi(uv) = \pi(u)\pi(v)$ for all $u, v \in T$, i.e., π is an automorphism of T . Applying ψ to both sides of (5), we obtain $\psi(X_u)\psi(X_v) = \beta(u, v)\psi(X_v)\psi(X_u)$. It follows that $\beta(\pi(u), \pi(v)) = \beta(u, v)$, for all $u, v \in T$. We have proved that the image of f is contained in $\text{Aut}(T, \beta)$. Conversely, suppose we have $\pi \in \text{Aut}(T, \beta)$. Observe that the algebra \mathcal{D} is generated by the elements X_{a_i} and X_{b_i} , $i = 1, \dots, r$. They satisfy $X_{a_i}^{\ell_i} = 1$, $X_{b_i}^{\ell_i} = 1$, and the commutation relations given by (5). Clearly, the free algebra modulo these relations has dimension $\leq \ell_1 \cdots \ell_r$. It follows that these relations are defining for the algebra \mathcal{D} . Since $\pi(a_i)$ has the same order as a_i , we can choose a scalar multiple $X'_{\pi(a_i)}$ of $X_{\pi(a_i)}$ such that $(X'_{\pi(a_i)})^{\ell_i} = 1$. Choose $X'_{\pi(b_i)}$ similarly. Since π preserves β , the elements $X'_{\pi(a_i)}$ and $X'_{\pi(b_i)}$ will satisfy the same commutation relations as X_{a_i} and X_{b_i} . It follows that there exists an automorphism ψ of \mathcal{D} sending X_{a_i} to $X'_{\pi(a_i)}$ and X_{b_i} to $X'_{\pi(b_i)}$, for all $i = 1, \dots, r$. By construction, $f(\psi)$ coincides with π on the elements a_i and b_i , $i = 1, \dots, r$. Hence $f(\psi) = \pi$. \square

Thus $W(\Gamma_0) = \text{Aut}(T, \beta)$. We are now ready to compute the Weyl group of any fine grading Γ on $M_n(\mathbb{F})$.

Theorem 2.8. Let $\Gamma = \Gamma_{\mathcal{M}}(\mathcal{D}, k)$ where \mathcal{D} is a matrix algebra with a division grading Γ_0 given by (4). Let T be the support of Γ_0 and let β be the nondegenerate alternating bicharacter $T \times T \rightarrow \mathbb{F}^\times$ determined by (5). Then

$$W(\Gamma) \cong T^{k-1} \rtimes (\text{Aut}(T, \beta) \times \text{Sym}(k))$$

where $\text{Aut}(T, \beta)$ and $\text{Sym}(k)$ act on T^{k-1} through identifying the latter with T^k/T , where T is imbedded into T^k diagonally.

Proof. Recall that any $\psi \in \text{Aut}(\Gamma)$ is given by (7) where $\Psi = PD$, P is a permutation matrix, $D = \text{diag}(d_1, \dots, d_k)$, and $\psi_0 \in \text{Aut}(\Gamma_0)$. Hence

$$\psi(E_{ij} \otimes d) = E_{\pi(i)\pi(j)} \otimes d_i \psi_0(d) d_j^{-1}. \quad (8)$$

Assume for a moment that $\psi \in \text{Stab}(\Gamma)$. Since all $\tilde{g}_i \tilde{g}_j^{-1}$, $i \neq j$, are distinct modulo T , we see by substituting $d = 1$ in (8) that $P = I$ and hence $D = \text{diag}(\lambda_1 \hat{d}, \dots, \lambda_k \hat{d})$ for some $\lambda_i \in \mathbb{F}^\times$ and nonzero homogeneous $\hat{d} \in \mathcal{D}$. We may simultaneously replace D by $\hat{D} = \text{diag}(\lambda_1, \dots, \lambda_k)$ and ψ_0 by $\hat{\psi}_0$ where $\hat{\psi}_0(d) = \hat{d} \psi_0(d) \hat{d}^{-1}$. Then taking $i = j$ in (8), we see that $\psi_0 \in \text{Stab}(\Gamma_0)$. Conversely, for $P = I$ and any $D = \text{diag}(\lambda_1, \dots, \lambda_k)$ and $\psi_0 \in \text{Stab}(\Gamma_0)$, Eq. (7) yields $\psi \in \text{Stab}(\Gamma)$. We have proved that $\text{Stab}(\Gamma)$ is isomorphic to $((\mathbb{F}^\times)^k / \mathbb{F}^\times) \times \text{Stab}(\Gamma_0)$. But by Proposition 2.7, we know that $\text{Stab}(\Gamma_0)$ is isomorphic to T via $t \mapsto \text{Ad}(X_t)$ where, for invertible $d \in \mathcal{D}$, $\text{Ad}(d) : \mathcal{D} \rightarrow \mathcal{D}$ is the inner automorphism $x \mapsto dx d^{-1}$.

For any $\psi_0 \in \text{Aut}(\Gamma_0)$, $\pi \in \text{Sym}(k)$, and nonzero homogeneous d_1, \dots, d_s , Eq. (7) defines a map $\psi : \mathcal{R} \rightarrow \mathcal{R}$ where $\Psi = PD$, P is the permutation matrix corresponding to π and $D = \text{diag}(d_1, \dots, d_k)$. Then for nonzero homogeneous d , the element $\psi(E_{ij} \otimes d)$ is also homogeneous. Since the degree of $\psi(E_{ij} \otimes d)$ does not depend on i and, for $i \neq j$, the element $E_{ij} \otimes d$ spans its homogeneous component, we conclude that ψ is an equivalence. It follows that the homomorphism $\text{Aut}(\Gamma) \rightarrow \text{Sym}(k)$ is onto and can be split by sending a permutation $\pi \in \text{Sym}(k)$ to the equivalence $\psi \in \text{Aut}(\Gamma)$ corresponding to $\psi_0 = \text{id}$ and $D = I$. Let \tilde{T} be the multiplicative group of nonzero homogeneous elements of \mathcal{D} , which is a central extension of T by \mathbb{F}^\times . The kernel K of the homomorphism $\text{Aut}(\Gamma) \rightarrow \text{Sym}(k)$ consists of the equivalences that correspond to $P = I$, so we have an epimorphism

$$f : (\tilde{T}^k / \mathbb{F}^\times) \rtimes \text{Aut}(\Gamma_0) \rightarrow K.$$

The kernel K_0 of f consists of the elements of the form $((d, \dots, d) \mathbb{F}^\times, \text{Ad}(d^{-1}))$, $d \in \tilde{T}$.

Let $N = A \times \text{Stab}(\Gamma_0)$ where $A \subset \tilde{T}^k / \mathbb{F}^\times$ consists of the elements of the form $(\lambda_1 d, \dots, \lambda_k d) \mathbb{F}^\times$ where $d \in \tilde{T}$ and $\lambda_i \in \mathbb{F}^\times$. Then N is a normal subgroup of $(\tilde{T}^k / \mathbb{F}^\times) \rtimes \text{Aut}(\Gamma_0)$. Clearly, $N \supset K_0$, and f maps N onto $\text{Stab}(\Gamma)$. It follows that

$$\begin{aligned} K / \text{Stab}(\Gamma) &\cong ((\tilde{T}^k / \mathbb{F}^\times) \rtimes \text{Aut}(\Gamma_0)) / (A \times \text{Stab}(\Gamma_0)) \\ &\cong ((\tilde{T}^k / \mathbb{F}^\times) / A) \rtimes (\text{Aut}(\Gamma_0) / \text{Stab}(\Gamma_0)). \end{aligned}$$

Now $(\tilde{T}^k / \mathbb{F}^\times) / A \cong T^k / T$ where T is imbedded into T^k diagonally. Also, we have $\text{Aut}(\Gamma_0) / \text{Stab}(\Gamma_0) \cong \text{Aut}(T, \beta)$ by Proposition 2.7. Hence

$$\text{Aut}(\Gamma) / \text{Stab}(\Gamma) \cong ((T^k / T) \rtimes \text{Aut}(T, \beta)) \rtimes \text{Sym}(k).$$

It remains to observe that the actions of $\text{Aut}(T, \beta)$ and of $\text{Sym}(k)$ on T^k / T commute with each other. \square

Recall that $U(\Gamma)$ is isomorphic to $\mathbb{Z}^{k-1} \times T$. To describe the action of $W(\Gamma)$ on $U(\Gamma)$, it is convenient to realize them as follows:

$$W(\Gamma) = (T^k/T) \rtimes (\text{Aut}(T, \beta) \times \text{Sym}(k)), \quad (9)$$

$$U(\Gamma) = \mathbb{Z}_0^k \times T, \quad (10)$$

where \mathbb{Z}_0^k is the subgroup of \mathbb{Z}^k consisting of all $\underline{x} = (x_1, \dots, x_k)$ such that $\sum_i x_i = 0$ (in other words, the elements \tilde{g}_i are identified with the standard basis in \mathbb{Z}^k).

Corollary 2.9 (Corollary of the proof). Writing $W(\Gamma)$ as in (9) and $U(\Gamma)$ as in (10), the action of $W(\Gamma)$ on $U(\Gamma)$ is the following:

- $(t_1, \dots, t_k) \in T^k$ acts by sending (\underline{x}, t) to $(\underline{x}, t \prod_i t_i^{x_i})$, and this action of T^k factors through T^k/T ;
- $\alpha \in \text{Aut}(T, \beta)$ acts by sending (\underline{x}, t) to $(\underline{x}, \alpha(t))$;
- $\pi \in \text{Sym}(k)$ acts on (\underline{x}, t) by permuting the components of \underline{x} . \square

Remark 2.10. We have also proved that $\text{Stab}(\Gamma) = \text{Diag}(\Gamma) \cong (\mathbb{F}^\times)^{k-1} \times T$. This is a special case of [BK10, Proposition 2.8], where $\text{Stab}(\Gamma)$ is computed for any abelian group grading Γ on $M_n(\mathbb{F})$.

Remark 2.11. The group $\text{Aut}(T, \beta)$ can be explicitly computed as follows. First, decompose T into primary components: $T = \prod_i T_i$ where T_i is a q_i -group, q_i is a prime, $q_i \neq \text{char } \mathbb{F}$. Then T_i are β -orthogonal, so we have

$$\text{Aut}(T, \beta) = \prod_i \text{Aut}(T_i, \beta_i) \quad \text{where } \beta_i = \beta|_{T_i \times T_i}.$$

So it is sufficient to consider the case when T is a q -group. Then (3) yields

$$T \cong ((\mathbb{Z}/q^{\alpha_1}\mathbb{Z}) \times (\mathbb{Z}/q^{\alpha_1}\mathbb{Z}))^{m_1} \times \dots \times ((\mathbb{Z}/q^{\alpha_f}\mathbb{Z}) \times (\mathbb{Z}/q^{\alpha_f}\mathbb{Z}))^{m_f}$$

where $\alpha_1 < \dots < \alpha_f$ and $m_i > 0$. Let $m = m_1 + \dots + m_f$. Then $\text{Aut}(T, \beta)$ can be identified with the group of $(2m \times 2m)$ -matrices A of the following form: A is partitioned into blocks A_{ij} of sizes $2m_i \times 2m_j$, $i, j = 1, \dots, f$, the entries of A_{ij} are integers modulo q^{α_i} , $A_{ij} \equiv 0 \pmod{q^{\alpha_i - \alpha_j}}$ for all $i > j$, and ${}^t A J A \equiv J \pmod{q^{\alpha_f}}$ where ${}^t A$ is the transpose of A and J is the diagonal sum of blocks $q^{\alpha_f - \alpha_i} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{\oplus m_i}$, $i = 1, \dots, f$.

3. The algebra of octonions

In this section, the Weyl groups of the fine gradings on the algebra of octonions will be computed. The Cayley algebra \mathbb{C} , or algebra of octonions, over \mathbb{F} is the unique, up to isomorphism, eight-dimensional unital composition algebra (recall that we are assuming \mathbb{F} algebraically closed). There exists a nondegenerate quadratic form (the norm) $n: \mathbb{C} \rightarrow \mathbb{F}$ such that $n(xy) = n(x)n(y)$ for all $x, y \in \mathbb{C}$. Here the norm being nondegenerate means that its polar form: $n(x, y) = n(x+y) - n(x) - n(y)$ is a nondegenerate symmetric bilinear form. Note that $n(x, x) = 2n(x)$.

The next result summarizes some of the well-known properties of this algebra (see [KMRT98, Chapter VIII] and [ZSSS82, Chapter 2]):

Proposition 3.1. Let \mathbb{C} be the Cayley algebra over \mathbb{F} . Then:

- 1) Any $x \in \mathbb{C}$ satisfies the degree 2 Cayley–Hamilton equation:

$$x^2 - n(x, 1)x + n(x)1 = 0.$$

Table 1

Multiplication table of the Cayley algebra.

	e_1	e_2	u_1	u_2	u_3	v_1	v_2	v_3
e_1	e_1	0	u_1	u_2	u_3	0	0	0
e_2	0	e_2	0	0	0	v_1	v_2	v_3
u_1	0	u_1	0	v_3	$-v_2$	$-e_1$	0	0
u_2	0	u_2	$-v_3$	0	v_1	0	$-e_1$	0
u_3	0	u_3	v_2	$-v_1$	0	0	0	$-e_1$
v_1	v_1	0	$-e_2$	0	0	0	u_3	$-u_2$
v_2	v_2	0	0	$-e_2$	0	$-u_3$	0	u_1
v_3	v_3	0	0	0	$-e_2$	u_2	$-u_1$	0

- 2) The map $x \mapsto \bar{x} = n(x, 1)1 - x$ is an involution, called the standard conjugation, of \mathbb{C} and for any $x, y, z \in \mathbb{C}$, we have $x\bar{x} = \bar{x}x = n(x)1$ and $n(xy, z) = n(y, \bar{x}z) = n(x, z\bar{y})$.
- 3) There is a “good basis” $\{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ of \mathbb{C} consisting of isotropic elements, such that $n(e_1, e_2) = n(u_i, v_i) = 1$ for any $i = 1, 2, 3$ and $n(e_r, u_i) = n(e_r, v_i) = n(u_i, u_j) = n(u_i, v_j) = n(v_i, v_j) = 0$ for any $r = 1, 2$ and $1 \leq i \neq j \leq 3$, whose multiplication table is shown in Table 1.

The linear map $t(x) = n(x, 1)$ is called the *trace*. A crucial step in the classification of fine gradings on \mathbb{C} [Eld98] is the fact that, for any grading $\mathbb{C} = \bigoplus_{g \in G} \mathbb{C}_g$, we have $t(\mathbb{C}_g \mathbb{C}_h) = 0$ unless $gh = e$.

3.1. Fine gradings on the algebra of octonions

A “good basis” of the Cayley algebra \mathbb{C} gives a \mathbb{Z}^2 -grading with

$$\begin{aligned}\mathbb{C}_{(0,0)} &= \mathbb{F}e_1 \oplus \mathbb{F}e_2, \\ \mathbb{C}_{(1,0)} &= \mathbb{F}u_1, & \mathbb{C}_{(-1,0)} &= \mathbb{F}v_1, \\ \mathbb{C}_{(0,1)} &= \mathbb{F}u_2, & \mathbb{C}_{(0,-1)} &= \mathbb{F}v_2, \\ \mathbb{C}_{(1,1)} &= \mathbb{F}v_3, & \mathbb{C}_{(-1,-1)} &= \mathbb{F}u_3.\end{aligned}$$

This is called the *Cartan grading* of the Cayley algebra. It is fine, and \mathbb{Z}^2 is its universal group.

Let \mathcal{Q} be a four-dimensional subalgebra of the Cayley algebra \mathbb{C} such that $n|_{\mathcal{Q}}$ is nondegenerate, and let u be any element in $\mathbb{C} \setminus \mathcal{Q}$ with $n(u) = \alpha \neq 0$. Then $\mathbb{C} = \mathcal{Q} \oplus \mathcal{Q}u$ and we get:

$$\begin{aligned}n(a + bu) &= n(a) + \alpha n(b), \\ (a + bu)(c + du) &= (ac - \alpha \bar{a}b) + (da + b\bar{c})u,\end{aligned}$$

for any $a, b, c, d \in \mathcal{Q}$. Then \mathbb{C} is said to be obtained from \mathcal{Q} by means of the *Cayley–Dickson doubling process* and we write $\mathbb{C} = \mathcal{CD}(\mathcal{Q}, \alpha)$. This gives a \mathbb{Z}_2 -grading on \mathbb{C} with $\mathbb{C}_0 = \mathcal{Q}$ and $\mathbb{C}_1 = \mathcal{Q}u$.

The subalgebra \mathcal{Q} above is a quaternion subalgebra which in turn can be obtained from a two-dimensional subalgebra \mathcal{K} through the same process $\mathcal{Q} = \mathcal{CD}(\mathcal{K}, \beta) = \mathcal{K} \oplus \mathcal{K}v$, and this gives a \mathbb{Z}_2 -grading of \mathcal{Q} and hence a \mathbb{Z}_2^2 -grading of $\mathbb{C} = \mathcal{K} \oplus \mathcal{K}v \oplus \mathcal{K}u \oplus (\mathcal{K}v)u$. We write here $\mathcal{Q} = \mathcal{CD}(\mathcal{K}, \beta, \alpha)$.

If $\text{char } \mathbb{F} \neq 2$, then \mathcal{K} can be obtained in turn from the ground field: $\mathcal{K} = \mathcal{CD}(\mathbb{F}, \gamma)$, and a \mathbb{Z}_2^3 -grading on \mathbb{C} appears. Here we write $\mathbb{C} = \mathcal{CD}(\mathbb{F}, \gamma, \beta, \alpha)$.

These gradings by \mathbb{Z}_2^r , $r = 1, 2, 3$, will be called *gradings induced by the Cayley–Dickson doubling process*. The groups \mathbb{Z}_2^r are their universal groups. Since there is a unique d -dimensional unital composition algebra for each $d = 2, 4, 8$, these \mathbb{Z}_2^r -gradings, $r = 1, 2, 3$, are unique up to equivalence.

The classification of fine gradings on \mathbb{C} was obtained in [Eld98]:

Theorem 3.2. *Let Γ be a fine abelian group grading on the Cayley algebra \mathbb{C} over an algebraically closed field \mathbb{F} . Then Γ is equivalent either to the Cartan grading or to the \mathbb{Z}_2^3 -grading induced by the Cayley–Dickson doubling process. The latter grading does not occur if $\text{char } \mathbb{F} = 2$. \square*

3.2. Cartan grading

Let S be the vector subspace spanned by $(1, 1, 1)$ in \mathbb{R}^3 and consider the two-dimensional real vector space $E = \mathbb{R}^3/S$. Take the elements

$$\epsilon_1 = (1, 0, 0) + S, \quad \epsilon_2 = (0, 1, 0) + S, \quad \epsilon_3 = (0, 0, 1) + S.$$

The subgroup $G = \mathbb{Z}\epsilon_1 + \mathbb{Z}\epsilon_2 + \mathbb{Z}\epsilon_3$ is isomorphic to \mathbb{Z}^2 , and we may think of the Cartan grading Γ of the Cayley algebra \mathbb{C} as the grading in which

$$\begin{aligned} \deg(e_1) &= 0 = \deg(e_2), \\ \deg(u_i) &= \epsilon_i = -\deg(v_i), \quad i = 1, 2, 3. \end{aligned}$$

Then $\text{Supp } \Gamma = \{0\} \cup \{\pm\epsilon_i \mid i = 1, 2, 3\}$. The set

$$\Phi := (\text{Supp } \Gamma \cup \{\alpha + \beta \mid \alpha, \beta \in \text{Supp } \Gamma, \alpha \neq \pm\beta\}) \setminus \{0\}$$

is the root system of type G_2 (although presented in a slightly different way from [Bou02, Chapter VI.4.13]).

Identifying the Weyl group $W(\Gamma)$ with a subgroup of $\text{Aut}(G)$, and this with a subgroup of $GL(E)$, we have:

$$\begin{aligned} W(\Gamma) &\subset \{\mu \in \text{Aut}(G) \mid \mu(\text{Supp } \Gamma) = \text{Supp } \Gamma\} \\ &\subset \{\mu \in GL(E) \mid \mu(\Phi) = \Phi\} =: \text{Aut } \Phi. \end{aligned}$$

The latter group is the automorphism group of the root system Φ , which coincides with its Weyl group.

If $\text{char } \mathbb{F} \neq 2, 3$, then we can work with the Lie algebra $\text{Der}(\mathbb{C})$ and prove the next theorem using known results on the simple Lie algebra of type G_2 in [Sel67]. The proof below works directly with the Cartan grading on the Cayley algebra and is valid in any characteristic.

Theorem 3.3. *Let Γ be the Cartan grading on the Cayley algebra over an algebraically closed field. Identify $\text{Supp } \Gamma \setminus \{0\}$ with the short roots in the root system Φ of type G_2 . Then $W(\Gamma) = \text{Aut } \Phi$.*

Proof. The group $\text{Aut } \Phi$ is the dihedral group of order 12. Now the order 3 automorphism of \mathbb{C} :

$$\tau : e_j \mapsto e_j, \quad u_i \mapsto u_{i+1}, \quad v_i \mapsto v_{i+1}, \quad (11)$$

for $j = 1, 2$ and $i = 1, 2, 3$ (modulo 3), belongs to $\text{Aut}(\Gamma)$, and its projection into $W(\Gamma)$ permutes cyclically the ϵ_i 's. Also, the order two automorphisms:

$$\begin{aligned} \varphi_1 : e_1 &\leftrightarrow e_2, \quad u_i \leftrightarrow v_i \quad (i = 1, 2, 3), \\ \varphi_2 : e_j &\mapsto e_j \quad (j = 1, 2), \quad u_1 \mapsto -u_1, \quad u_2 \leftrightarrow u_3, \quad v_1 \mapsto -v_1, \quad v_2 \leftrightarrow v_3, \end{aligned}$$

belong to $\text{Aut}(\Gamma)$, and their projections into $W(\Gamma)$ generate a subgroup of order 4. Therefore, the order of $W(\Gamma)$ is at least $4 \times 3 = 12$, and hence $W(\Gamma)$ is the whole $\text{Aut } \Phi$. \square

Remark 3.4. We have $\text{Stab}(\Gamma) = \text{Diag}(\Gamma)$. It is a maximal torus in the algebraic group $\text{Aut}(\mathbb{C})$.

3.3. \mathbb{Z}_2^3 -grading

Recall that this grading occurs only if $\text{char } \mathbb{F} \neq 2$. We fix the following notation: let

$$c_1 = (\bar{1}, \bar{0}, \bar{0}), \quad c_2 = (\bar{0}, \bar{1}, \bar{0}), \quad c_3 = (\bar{0}, \bar{0}, \bar{1})$$

be the standard basis of \mathbb{Z}_2^3 . The Cayley algebra \mathbb{C} is obtained by repeated application of the Cayley–Dickson doubling process:

$$\mathcal{K} = \mathbb{F} \oplus \mathbb{F}w_1, \quad \mathcal{Q} = \mathcal{K} \oplus \mathcal{K}w_2, \quad \mathbb{C} = \mathcal{Q} \oplus \mathcal{Q}w_3,$$

with $w_i^2 = 1$ for $i = 1, 2, 3$. (One may take $w_1 = e_1 - e_2$, $w_2 = u_1 - v_1$ and $w_3 = u_2 - v_2$.) Setting

$$\deg w_i = c_i, \quad i = 1, 2, 3, \tag{12}$$

gives the \mathbb{Z}_2^3 -grading Γ induced by the Cayley–Dickson doubling process.

Theorem 3.5. Let Γ be the \mathbb{Z}_2^3 -grading on the Cayley algebra as in (12) over an algebraically closed field of characteristic different from 2. Then $W(\Gamma) = \text{Aut}(\mathbb{Z}_2^3) \cong GL_3(2)$.

Proof. Given any $\mu \in \text{Aut}(\mathbb{Z}_2^3)$, take $\tilde{w}_i \in \mathbb{C}_{\mu(c_i)}$ with $\tilde{w}_i^2 = 1$. Then \mathbb{C} is obtained by repeated application of the Cayley–Dickson doubling process:

$$\tilde{\mathcal{K}} = \mathbb{F} \oplus \mathbb{F}\tilde{w}_1, \quad \tilde{\mathcal{Q}} = \tilde{\mathcal{K}} \oplus \tilde{\mathcal{K}}\tilde{w}_2, \quad \mathbb{C} = \tilde{\mathcal{Q}} \oplus \tilde{\mathcal{Q}}\tilde{w}_3,$$

and hence there is a unique automorphism $\varphi \in \text{Aut } \mathbb{C}$ such that $\varphi(w_i) = \tilde{w}_i$ for $i = 1, 2, 3$. Then φ belongs to $\text{Aut}(\Gamma)$, and its projection into $W(\Gamma)$ is precisely μ . This shows that $W(\Gamma)$ fills the whole $\text{Aut}(\mathbb{Z}_2^3)$. \square

Remark 3.6. As any $\varphi \in \text{Stab}(\Gamma)$ multiplies each w_i , $i = 1, 2, 3$, by either 1 or -1 , we see that $\text{Stab}(\Gamma) = \text{Diag}(\Gamma)$ is isomorphic to \mathbb{Z}_2^3 . Therefore, the group $\text{Aut}(\Gamma)$ is a (non-split) extension of \mathbb{Z}_2^3 by $W(\Gamma) \cong GL_3(2)$. This group $\text{Aut}(\Gamma)$, in its irreducible seven-dimensional representation given by the trace zero elements in \mathbb{C} , is precisely the group used in [Wil09] to give a nice construction of the compact real form of the Lie algebra of type G_2 .

4. The Albert algebra

In this section, the Weyl groups of the fine gradings on the Albert algebra will be computed. We assume $\text{char } \mathbb{F} \neq 2$ throughout this section.

Let \mathbb{C} be the Cayley algebra. The *Albert algebra* is the algebra of Hermitian 3×3 -matrices over \mathbb{C} :

$$\begin{aligned} \mathcal{A} = \mathcal{H}_3(\mathbb{C}, *) &= \left\{ \begin{pmatrix} \alpha_1 & \bar{a}_3 & a_2 \\ a_3 & \alpha_2 & \bar{a}_1 \\ \bar{a}_2 & a_1 & \alpha_3 \end{pmatrix} \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}, a_1, a_2, a_3 \in \mathbb{C} \right\} \\ &= \mathbb{F}E_1 \oplus \mathbb{F}E_2 \oplus \mathbb{F}E_3 \oplus \iota_1(\mathbb{C}) \oplus \iota_2(\mathbb{C}) \oplus \iota_3(\mathbb{C}), \end{aligned}$$

where

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\iota_1(a) = 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{a} \\ 0 & a & 0 \end{pmatrix}, \quad \iota_2(a) = 2 \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ \bar{a} & 0 & 0 \end{pmatrix}, \quad \iota_3(a) = 2 \begin{pmatrix} 0 & \bar{a} & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with (commutative) multiplication given by $XY = \frac{1}{2}(X \cdot Y + Y \cdot X)$, where $X \cdot Y$ denotes the usual product of matrices X and Y . Then E_i are orthogonal idempotents with $E_1 + E_2 + E_3 = 1$. The rest of the products are as follows:

$$E_i \iota_i(a) = 0, \quad E_{i+1} \iota_i(a) = \frac{1}{2} \iota_i(a) = E_{i+2} \iota_i(a),$$

$$\iota_i(a) \iota_{i+1}(b) = \iota_{i+2}(\bar{a}b), \quad \iota_i(a) \iota_i(b) = 2n(a, b)(E_{i+1} + E_{i+2}), \quad (13)$$

for any $a, b \in \mathbb{C}$, with $i = 1, 2, 3$ taken modulo 3. (This convention about indices will be used without further mention.)

For the main properties of the Albert algebra the reader may consult [Jac68]. It is the only exceptional simple Jordan algebra over \mathbb{F} . Any $X \in \mathcal{A}$ satisfies the degree 3 Cayley–Hamilton equation:

$$X^3 - T(X)X^2 + S(X)X - N(X)1 = 0,$$

where the linear form T is called the *trace* and the cubic form N is called the *norm* of the Albert algebra. A crucial step in the classification of fine gradings on \mathcal{A} [EK12] is the fact that, for any grading $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$, we have $T(\mathcal{A}_g \mathcal{A}_h) = 0$ unless $gh = e$. We also note that any automorphism φ of \mathbb{C} extends to \mathcal{A} by setting $\varphi(E_i) = E_i$, $\varphi(\iota_i(x)) = \iota_i(\varphi(x))$, for all $x \in \mathbb{C}$ and $i = 1, 2, 3$.

4.1. Fine gradings on the Albert algebra

First we describe the gradings in question as they are presented in [EK12].

Cartan grading: Consider the following elements in $\mathbb{Z}^4 = \mathbb{Z}^2 \times \mathbb{Z}^2$:

$$a_1 = (1, 0, 0, 0), \quad a_2 = (0, 1, 0, 0), \quad a_3 = (-1, -1, 0, 0),$$

$$g_1 = (0, 0, 1, 0), \quad g_2 = (0, 0, 0, 1), \quad g_3 = (0, 0, -1, -1).$$

Then $a_1 + a_2 + a_3 = 0 = g_1 + g_2 + g_3$. Take a “good basis” of the Cayley algebra \mathbb{C} . Recall that the assignment

$$\deg e_1 = \deg e_2 = 0, \quad \deg u_i = g_i = -\deg v_i$$

gives the Cartan grading on \mathbb{C} .

Now, the assignment

$$\deg E_i = 0,$$

$$\deg \iota_i(e_1) = a_i = -\deg \iota_i(e_2),$$

$$\deg \iota_i(u_i) = g_i = -\deg \iota_i(v_i),$$

$$\deg \iota_i(u_{i+1}) = a_{i+2} + g_{i+1} = -\deg \iota_i(v_{i+1}),$$

$$\deg \iota_i(u_{i+2}) = -a_{i+1} + g_{i+2} = -\deg \iota_i(v_{i+2}),$$

where $i = 1, 2, 3$, gives a \mathbb{Z}^4 -grading on the Albert algebra \mathcal{A} . (To see this, it suffices to look at the first component of $\mathbb{Z}^2 \times \mathbb{Z}^2$, and by the cyclic symmetry of the product, it is enough to check that $\deg(\iota_3(\bar{x}\bar{y})) = \deg \iota_1(x) + \deg \iota_2(y)$ for any x, y in the “good basis” of \mathcal{C} , and this is straightforward.)

This \mathbb{Z}^4 -grading will be called the *Cartan grading* on the Albert algebra. It is fine, and \mathbb{Z}^4 is its universal group (see [EK12]).

\mathbb{Z}_2^5 -grading: Recall the \mathbb{Z}_2^3 -grading (12) on the Cayley algebra \mathcal{C} induced by the Cayley–Dickson doubling process. Then \mathcal{A} is obviously \mathbb{Z}_2^5 -graded as follows:

$$\deg E_i = (\bar{0}, \bar{0}, \bar{0}, \bar{0}, \bar{0}), \quad i = 1, 2, 3,$$

$$\deg \iota_1(x) = (\bar{1}, \bar{0}, \deg x),$$

$$\deg \iota_2(x) = (\bar{0}, \bar{1}, \deg x),$$

$$\deg \iota_3(x) = (\bar{1}, \bar{1}, \deg x),$$

for homogeneous elements $x \in \mathcal{C}$.

This grading will be referred to as the \mathbb{Z}_2^5 -grading on the Albert algebra. It is fine, and \mathbb{Z}_2^5 is its universal group (see [EK12]).

$\mathbb{Z} \times \mathbb{Z}_2^3$ -grading: Take an element $\mathbf{i} \in \mathbb{F}$ with $\mathbf{i}^2 = -1$ and consider the following elements in \mathcal{A} :

$$E = E_1,$$

$$\tilde{E} = 1 - E = E_2 + E_3,$$

$$v(a) = \mathbf{i}\iota_1(a) \quad \text{for all } a \in \mathcal{C}_0,$$

$$v_{\pm}(x) = \iota_2(x) \pm \mathbf{i}\iota_3(\bar{x}) \quad \text{for all } x \in \mathcal{C},$$

$$S^{\pm} = E_3 - E_2 \pm \frac{\mathbf{i}}{2}\iota_1(1),$$

where $\mathcal{C}_0 = \{a \in \mathcal{C} \mid t(a) = 0\}$. The above elements span \mathcal{A} , and (13) translates to:

$$E\tilde{E} = 0, \quad ES^{\pm} = 0, \quad Ev(a) = 0, \quad Ev_{\pm}(x) = \frac{1}{2}v_{\pm}(x),$$

$$\tilde{E}S^{\pm} = S^{\pm}, \quad \tilde{E}v(a) = v(a), \quad \tilde{E}v_{\pm}(x) = \frac{1}{2}v_{\pm}(x),$$

$$S^+S^- = 2\tilde{E}, \quad S^{\pm}v(a) = 0, \quad S^{\pm}v_{\mp}(x) = v_{\pm}(x), \quad S^{\pm}v_{\pm}(x) = 0,$$

$$v(a)v(b) = -2n(a, b)\tilde{E}, \quad v(a)v_{\pm}(x) = \pm v_{\pm}(xa),$$

$$v_{\pm}(x)v_{\pm}(y) = 2n(x, y)S^{\pm}, \quad v_{+}(x)v_{-}(y) = 2n(x, y)(2E + \tilde{E}) + v(\bar{x}y - \bar{y}x),$$

for all $x, y \in \mathcal{C}$ and $a, b \in \mathcal{C}_0$.

There appears a \mathbb{Z} -grading on \mathcal{A} :

$$\mathcal{A} = \mathcal{A}_{-2} \oplus \mathcal{A}_{-1} \oplus \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2,$$

Table 2

Multiplication table of the Okubo algebra.

	e_1	e_2	u_1	v_1	u_2	v_2	u_3	v_3
e_1	e_2	0	0	$-v_3$	0	$-v_1$	0	$-v_2$
e_2	0	e_1	$-u_3$	0	$-u_1$	0	$-u_2$	0
u_1	$-u_2$	0	v_1	0	$-v_3$	0	0	$-e_1$
v_1	0	$-v_2$	0	u_1	0	$-u_3$	$-e_2$	0
u_2	$-u_3$	0	0	$-e_1$	v_2	0	$-v_1$	0
v_2	0	$-v_3$	$-e_2$	0	0	u_2	0	$-u_1$
u_3	$-u_1$	0	$-v_2$	0	0	$-e_1$	v_3	0
v_3	0	$-v_1$	0	$-u_2$	$-e_2$	0	0	u_3

with $\mathcal{A}_{\pm 2} = \mathbb{F}S^{\pm}$, $\mathcal{A}_{\pm 1} = v_{\pm}(\mathbb{C})$, and $\mathcal{A}_0 = \mathbb{F}E \oplus (\mathbb{F}\tilde{E} \oplus v(\mathbb{C}_0))$. The \mathbb{Z}_2^3 -grading on \mathbb{C} combines with this \mathbb{Z} -grading to give a $\mathbb{Z} \times \mathbb{Z}_2^3$ -grading as follows:

$$\begin{aligned}
 \deg S^{\pm} &= (\pm 2, \bar{0}, \bar{0}, \bar{0}), \\
 \deg v_{\pm}(x) &= (\pm 1, \deg x), \\
 \deg E &= 0 = \deg \tilde{E}, \\
 \deg v(a) &= (0, \deg a),
 \end{aligned} \tag{14}$$

for homogeneous elements $x \in \mathbb{C}$ and $a \in \mathbb{C}_0$.

This grading will be referred to as the $\mathbb{Z} \times \mathbb{Z}_2^3$ -grading on the Albert algebra. It is fine, and $\mathbb{Z} \times \mathbb{Z}_2^3$ is its universal group (see [EK12]).

\mathbb{Z}_3^3 -grading: Consider the order 3 automorphism τ of \mathbb{C} in (11), and the new multiplication defined on \mathbb{C} by:

$$x * y := \tau(\bar{x})\tau^2(\bar{y}) \quad \text{for all } x, y \in \mathbb{C}.$$

Then $n(x * y) = n(x)n(y)$ for all x, y , since τ preserves the norm. Moreover, for all $x, y, z \in \mathbb{C}$, we have:

$$\begin{aligned}
 n(x * y, z) &= n(\tau(\bar{x})\tau^2(\bar{y}), z) \\
 &= n(\tau(\bar{x}), z\tau^2(y)) \\
 &= n(\bar{x}, \tau^2(z)\tau(y)) \\
 &= n(x, \tau(\bar{y})\tau^2(\bar{z})) \\
 &= n(x, y * z).
 \end{aligned}$$

Hence $(\mathbb{C}, *, n)$ is a symmetric composition algebra (see [Eld09] or [KMRT98, Chapter VIII]). Actually, $(\mathbb{C}, *)$ is the Okubo algebra over \mathbb{F} . Its multiplication table is shown in Table 2.

This Okubo algebra is \mathbb{Z}_3^2 -graded by setting $\deg e_1 = (\bar{1}, \bar{0})$ and $\deg u_1 = (\bar{0}, \bar{1})$, with the degrees of the remaining elements being uniquely determined.

Assume now that $\text{char } \mathbb{F} \neq 3$. Then this \mathbb{Z}_3^2 -grading is determined by two commuting order 3 automorphisms $\varphi_1, \varphi_2 \in \text{Aut}(\mathbb{C}, *)$:

$$\varphi_1(e_1) = \omega e_1, \quad \varphi_1(u_1) = u_1 \quad \text{and} \quad \varphi_2(e_1) = e_1, \quad \varphi_2(u_1) = \omega u_1,$$

where ω is a primitive third root of unity in \mathbb{F} .

Define $\tilde{\iota}_i(x) = \iota_i(\tau^i(x))$ for all $i = 1, 2, 3$ and $x \in \mathbb{C}$. Then (13) translates to:

$$\begin{aligned} E_i \tilde{\iota}_i(x) &= 0, & E_{i+1} \tilde{\iota}_i(x) &= \frac{1}{2} \tilde{\iota}_i(x) = E_{i+2} \tilde{\iota}_i(x), \\ \tilde{\iota}_i(x) \tilde{\iota}_{i+1}(y) &= \tilde{\iota}_{i+2}(x * y), & \tilde{\iota}_i(x) \tilde{\iota}_i(y) &= 2n(x, y)(E_{i+1} + E_{i+2}), \end{aligned}$$

for $i = 1, 2, 3$ and $x, y \in \mathbb{C}$.

The commuting order 3 automorphisms φ_1, φ_2 of $(\mathbb{C}, *)$ extend to commuting order 3 automorphisms of \mathcal{A} (which will be denoted by the same symbols) as follows:

$$\varphi_j(E_i) = E_i, \quad \varphi_j(\tilde{\iota}_i(x)) = \tilde{\iota}_i(\varphi_j(x)) \quad \text{for all } x \in \mathbb{C} \text{ and } i = 1, 2, 3, \quad (15)$$

where $j = 1, 2$. On the other hand, the linear map $\varphi_3 : \mathcal{A} \rightarrow \mathcal{A}$ defined by

$$\varphi_3(E_i) = E_{i+1}, \quad \varphi_3(\tilde{\iota}_i(x)) = \tilde{\iota}_{i+1}(x) \quad \text{for all } x \in \mathbb{C} \text{ and } i = 1, 2, 3, \quad (16)$$

is another order 3 automorphism, which commutes with φ_1 and φ_2 . The subgroup of $\text{Aut } \mathcal{A}$ generated by $\varphi_1, \varphi_2, \varphi_3$ is isomorphic to \mathbb{Z}_3^3 and induces a \mathbb{Z}_3^3 -grading on \mathcal{A} whose components are all 1-dimensional. This grading is obviously fine, and \mathbb{Z}_3^3 is its universal group. The nonzero homogeneous elements are invertible (in the Jordan sense), and any three of them whose degrees are independent in \mathbb{Z}_3^3 generate the Albert algebra (see [EK12, §7.3]).

The (unique up to equivalence) grading induced by $\varphi_1, \varphi_2, \varphi_3$ will be referred to as the \mathbb{Z}_3^3 -grading on the Albert algebra.

The classification of fine gradings on the Albert algebra was obtained in [DM09] for $\text{char } \mathbb{F} = 0$ and in [EK12] for $\text{char } \mathbb{F} \neq 2$.

Theorem 4.1. *Let Γ be a fine abelian group grading on the Albert algebra \mathcal{A} over an algebraically closed field \mathbb{F} of characteristic different from 2. Then Γ is equivalent either to the Cartan grading, the \mathbb{Z}_2^5 -grading, the $\mathbb{Z} \times \mathbb{Z}_2^3$ -grading or the \mathbb{Z}_3^3 -grading. The last grading does not occur if $\text{char } \mathbb{F} = 3$. \square*

4.2. Cartan grading

The universal group is \mathbb{Z}^4 , which is contained in $E = \mathbb{R}^4$. Consider the following elements of \mathbb{Z}^4 :

$$\begin{aligned} \epsilon_0 &= \deg \iota_1(e_1) = a_1 = (1, 0, 0, 0), \\ \epsilon_1 &= \deg \iota_1(u_1) = g_1 = (0, 0, 1, 0), \\ \epsilon_2 &= \deg \iota_1(u_2) = a_3 + g_2 = (-1, -1, 0, 1), \\ \epsilon_3 &= \deg \iota_1(u_3) = -a_2 + g_3 = (0, -1, -1, -1). \end{aligned}$$

Note that ϵ_i , $0 \leq i \leq 3$, are linearly independent, but do not form a basis of \mathbb{Z}^4 . We have for instance $\deg \iota_2(e_1) = a_2 = \frac{1}{2}(-\epsilon_0 - \epsilon_1 - \epsilon_2 - \epsilon_3)$ and $\deg \iota_3(e_1) = \frac{1}{2}(-\epsilon_0 + \epsilon_1 + \epsilon_2 + \epsilon_3)$. Moreover, the supports of the Cartan grading Γ on each of the subspaces $\iota_i(\mathbb{C})$ are:

$$\begin{aligned} \text{Supp } \iota_1(\mathbb{C}) &= \{\pm \epsilon_i \mid 0 \leq i \leq 3\}, \\ \text{Supp } \iota_2(\mathbb{C}) &= \text{Supp } \iota_1(\mathbb{C}) (\iota_3(e_1) + \iota_3(e_2)) \\ &= \left\{ \frac{1}{2}(\pm \epsilon_0 \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3) \mid \text{even number of } + \text{ signs} \right\}, \end{aligned}$$

$$\begin{aligned}\text{Supp } \iota_3(\mathbb{C}) &= \text{Supp } \iota_1(\mathbb{C})(\iota_2(e_1) + \iota_2(e_2)) \\ &= \left\{ \frac{1}{2}(\pm\epsilon_0 \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3) \mid \text{odd number of } + \text{ signs} \right\}.\end{aligned}$$

Let us consider the following subset Φ of E :

$$\begin{aligned}\Phi &:= (\text{Supp } \Gamma \cup \{\alpha + \beta \mid \alpha, \beta \in \text{Supp } \iota_1(\mathbb{C}), \alpha \neq \pm\beta\}) \setminus \{0\} \\ &= \text{Supp } \iota_1(\mathbb{C}) \cup \text{Supp } \iota_2(\mathbb{C}) \cup \text{Supp } \iota_3(\mathbb{C}) \cup \{\pm\epsilon_i \pm \epsilon_j \mid 0 \leq i \neq j \leq 3\},\end{aligned}$$

which is the root system of type F_4 (see [Bou02, Chapter VI.4.9]). Note that ϵ_i , $i = 0, 1, 2, 3$, form an orthogonal basis of E relative to the unique (up to scalar) inner product that is invariant under the Weyl group of Φ .

Identifying the Weyl group $W(\Gamma)$ with a subgroup of $\text{Aut}(\mathbb{Z}^4)$, and this with a subgroup of $GL(E)$, we have:

$$\begin{aligned}W(\Gamma) &\subset \{\mu \in \text{Aut}(\mathbb{Z}^4) \mid \mu(\text{Supp } \Gamma) = \text{Supp } \Gamma\} \\ &\subset \{\mu \in GL(E) \mid \mu(\Phi) = \Phi\} =: \text{Aut } \Phi.\end{aligned}$$

The latter group is the automorphism group of the root system Φ , which coincides with its Weyl group.

If $\text{char } \mathbb{F} \neq 2, 3$, then we can work with the Lie algebra $\text{Der}(\mathcal{A})$ and prove the next theorem using known results on the simple Lie algebra of type F_4 in [Sel67]. The proof below works directly with the Cartan grading on the Albert algebra and is valid for $\text{char } \mathbb{F} \neq 2$.

Theorem 4.2. *Let Γ be the Cartan grading on the Albert algebra over an algebraically closed field of characteristic different from 2. Identify $\text{Supp } \Gamma \setminus \{0\}$ with the short roots in the root system Φ of type F_4 . Then $W(\Gamma) = \text{Aut } \Phi$.*

Proof. Let us obtain first some distinguished elements in $W(\Gamma)$.

1) The order 3 automorphism of \mathcal{A} : $\psi_{(123)}: E_i \mapsto E_{i+1}$, $\iota_i(x) \mapsto \iota_{i+1}(x)$, is in $\text{Aut}(\Gamma)$, and its projection $\mu_{(123)}$ into $W(\Gamma)$ permutes cyclically $\text{Supp } \iota_1(\mathbb{C})$, $\text{Supp } \iota_2(\mathbb{C})$ and $\text{Supp } \iota_3(\mathbb{C})$.

2) The order 2 automorphism of \mathcal{A} : $\psi_{(23)}: E_1 \mapsto E_1$, $E_2 \leftrightarrow E_3$, $\iota_1(x) \mapsto \iota_1(\bar{x})$, $\iota_2(x) \leftrightarrow \iota_3(\bar{x})$, also belongs to $\text{Aut}(\Gamma)$, and its projection $\mu_{(23)}$ sends ϵ_0 to $-\epsilon_0$ (as $\psi_{(23)}(\iota_1(e_1)) = \iota_1(e_2)$), and leaves invariant each ϵ_i for $i = 1, 2, 3$.

3) Now consider $\mathcal{C}l(\mathbb{C}, n)$, the Clifford algebra of \mathbb{C} (regarded as a quadratic space), and the group $\text{Spin}(\mathbb{C}, n) \subset \mathcal{C}l(\mathbb{C}, n)$. It is well-known that any $c \in \text{Spin}(\mathbb{C}, n)$ gives rise to an automorphism ψ_c of the Albert algebra fixing the idempotents E_1, E_2, E_3 — see e.g. [KMRT98]. Explicitly, ψ_c is defined by setting $\psi_c(\iota_1(z)) = \iota_1(\chi_c(z))$, $\psi_c(\iota_2(z)) = \iota_2(\rho_c^+(z))$ and $\psi_c(\iota_3(z)) = \iota_3(\rho_c^-(z))$ for all $z \in \mathbb{C}$, where $\chi_c(z) = c \cdot z \cdot c^{-1}$ and \cdot denotes multiplication in the Clifford algebra. If $c = x \cdot y$ with $x, y \in \mathbb{C}$, then

$$\rho_c^+(z) = (zy)\bar{x} \quad \text{and} \quad \rho_c^-(z) = \bar{x}(yz) \quad \text{for all } z \in \mathbb{C}. \quad (17)$$

Consider the elements

$$x = \frac{1}{\sqrt{2}}(e_1 + e_2 + u_1 + v_1) \quad \text{and} \quad y = \frac{\mathbf{i}}{\sqrt{2}}(e_1 - e_2 + u_1 - v_1).$$

Then $c = x \cdot y \in \text{Spin}(\mathbb{C}, n)$, since $n(x) = n(y) = 1$. Also note that $c^2 = -1$, as x and y are orthogonal, and so χ_c has order 2. Since $e_1 + e_2$ and y are also orthogonal, we compute:

$$\begin{aligned}
\chi_c(e_1 + e_2) &= c \cdot (e_1 + e_2) \cdot c^{-1} = -c \cdot (e_1 + e_2) \cdot c = -x \cdot (e_1 + e_2) \cdot x \cdot y^2 \\
&= -\frac{1}{2}(1 - (e_1 + e_2) \cdot (u_1 + v_1)) \cdot (e_1 + e_2 + u_1 + v_1) \\
&= -\frac{1}{2}(e_1 + e_2 + (e_1 + e_2)^2 \cdot (u_1 + v_1) + (u_1 + v_1) - (e_1 + e_2)) \\
&= -(u_1 + v_1).
\end{aligned}$$

A similar calculation shows that $\chi_c(e_1 - e_2) = -(u_1 - v_1)$. Hence we have:

$$\chi_c: e_1 \leftrightarrow -u_1, \quad e_2 \leftrightarrow -v_1, \quad u_2 \mapsto u_2, \quad u_3 \mapsto u_3, \quad v_2 \mapsto v_2, \quad v_3 \mapsto v_3.$$

Let us check that the associated automorphism ψ_c of \mathcal{A} is in $\text{Aut}(\Gamma)$. Since the action of ψ_c on $\iota_1(\mathbb{C})$ is given by χ_c , we already know that ψ_c permutes the homogeneous components of $\iota_1(\mathbb{C})$. Since every homogeneous element of $\iota_3(\mathbb{C})$ lies either in $\iota_1(\mathbb{C})\iota_2(e_1)$ or in $\iota_1(\mathbb{C})\iota_2(e_2)$, and every homogeneous element of $\iota_2(\mathbb{C})$ lies either in $\iota_1(\mathbb{C})\iota_3(e_1)$ or in $\iota_1(\mathbb{C})\iota_3(e_2)$, it remains to check that $\psi_c(\iota_i(e_j))$ is homogeneous for $i = 2, 3$ and $j = 1, 2$. Using (17), we compute:

$$\begin{aligned}
\psi_c(\iota_2(e_1)) &= \iota_2(\rho_c^+(e_1)) = \iota_2((e_1 y) \bar{x}) \\
&= \frac{i}{2} \iota_2((e_1(e_1 - e_2 + u_1 - v_1))(e_1 + e_2 - u_1 - v_1)) \\
&= \frac{i}{2} \iota_2((e_1 + u_1)(e_1 + e_2 - u_1 - v_1)) = \frac{i}{2} \iota_2(e_1 - u_1 + u_1 + e_1) = i \iota_2(e_1).
\end{aligned}$$

Similar calculations show that $\psi_c(\iota_2(e_2)) = -i \iota_2(e_2)$, $\psi_c(\iota_3(e_1)) = -i \iota_3(v_1)$, and $\psi_c(\iota_3(e_2)) = i \iota_3(u_1)$. Therefore, $\psi_c \in \text{Aut}(\Gamma)$, and its projection μ_c into $W(\Gamma)$ acts as follows: $e_0 \leftrightarrow e_1$, $e_2 \mapsto e_2$, $e_3 \mapsto e_3$.

4) Finally, the order 3 automorphism τ of \mathbb{C} given by (11) extends to an automorphism of \mathcal{A} fixing E_i via $\iota_i(x) \mapsto \iota_i(\tau(x))$ for all $x \in \mathbb{C}$ and $i = 1, 2, 3$. The projection of this automorphism into $W(\Gamma)$ is the 3-cycle $e_1 \mapsto e_2 \mapsto e_3 \mapsto e_1$.

Now we are ready to prove the theorem. Any $\mu \in \text{Aut } \Phi$ permutes the subsets $\text{Supp } \iota_i(\mathbb{C})$, as these are the only subsets S of $\text{Supp } \Gamma \setminus \{0\}$ such that for any $\delta \in S$, $S = \{\pm\delta\} \cup \{\gamma \in \text{Supp } \Gamma \setminus \{0\} \mid (\gamma, \delta) = 0\}$. Thus, composing with a suitable power of $\mu_{(123)}$ from 1), we may assume $\mu(\text{Supp } \iota_1(\mathbb{C})) = \text{Supp } \iota_1(\mathbb{C})$. But the group $\{\mu \in \text{Aut } \Phi \mid \mu(\text{Supp } \iota_1(\mathbb{C})) = \text{Supp } \iota_1(\mathbb{C})\}$ is isomorphic to $\mathbb{Z}_2^4 \rtimes \text{Sym}(4)$, consisting of the permutations of the e_i 's followed by multiplication of some of the e_i 's by -1 . This subgroup is generated by $\mu_{(23)}$ from 2), the transposition μ_c from 3), and the 3-cycle from 4). \square

Remark 4.3. We have $\text{Stab}(\Gamma) = \text{Diag}(\Gamma)$. It is a maximal torus in the algebraic group $\text{Aut}(\mathcal{A})$.

4.3. \mathbb{Z}_2^5 -grading

Write $\mathbb{Z}_2^5 = \mathbb{Z}_2^2 \times \mathbb{Z}_2^3$ where \mathbb{Z}_2^3 is generated by c_j , $j = 1, 2, 3$, as in (12). Then the \mathbb{Z}_2^5 -grading Γ is defined by setting

$$\deg \iota_1(1) = a, \quad \deg \iota_2(1) = b, \quad \deg \iota_3(w_j) = a + b + c_j, \quad j = 1, 2, 3, \quad (18)$$

where $\{a, b\}$ is the standard basis of \mathbb{Z}_2^2 .

Theorem 4.4. Let Γ be the \mathbb{Z}_2^5 -grading on the Albert algebra as in (18) over an algebraically closed field of characteristic different from 2. Let T be the subgroup of \mathbb{Z}_2^5 generated by c_j , $j = 1, 2, 3$. Then $W(\Gamma) = \{\mu \in \text{Aut}(\mathbb{Z}_2^5) \mid \mu(T) = T\}$.

Proof. Let K the subgroup generated by a and b . Then $\mathbb{Z}_2^5 = K \times T$. Identifying $\text{Aut}(\mathbb{Z}_2^5)$ with $GL_5(2)$, the stabilizer of T (as a set) consists of all matrices of the form $\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$. The automorphism $\psi_{(123)}$ defined in step 1) in the proof of Theorem 4.2 and the automorphism $\psi_{(12)}$ that is analogous to $\psi_{(23)}$ defined in step 2) belong to $\text{Aut}(\Gamma)$, and their projections into $W(\Gamma)$ act as follows:

$$\mu_{(123)} : a \mapsto b \mapsto a + b \mapsto a, \quad c_j \mapsto c_j, \quad j = 1, 2, 3,$$

$$\mu_{(12)} : a \leftrightarrow b, \quad c_j \mapsto c_j, \quad j = 1, 2, 3.$$

Therefore, the subgroup $\{\mu \in GL_5(2) \mid \mu(K) = K, \mu|_T = \text{id}\}$, which consists of matrices of the form $\begin{pmatrix} * & 0 \\ 0 & I \end{pmatrix}$, is contained in $W(\Gamma)$. (Note that this subgroup is the symmetric group on the elements a, b and $a + b$.)

Now, consider the subgroup $\{\mu \in GL_5(2) \mid \mu(T) = T, \mu|_K = \text{id}\}$, which consists of all matrices of the form $\begin{pmatrix} I & 0 \\ 0 & * \end{pmatrix}$. By Theorem 3.5, for any such μ , there is an automorphism φ of \mathcal{C} that belongs to $\text{Aut}(\Gamma_0)$, where Γ_0 is the \mathbb{Z}_2^3 -grading (12) on \mathcal{C} , such that the projection of φ into $W(\Gamma_0)$ coincides with $\mu|_T$. Then the automorphism of \mathcal{A} that fixes E_i and takes $\iota_i(x)$ to $\iota_i(\varphi(x))$, for all $x \in \mathcal{C}$ and $i = 1, 2, 3$, belongs to $\text{Aut}(\Gamma)$, and its projection into $W(\Gamma)$ coincides with μ . Hence the subgroup under consideration is contained in $W(\Gamma)$.

For any $h \in T$, consider the element $\mu \in GL_5(2)$ such that $\mu(a) = a$, $\mu(b) = b + h$, and $\mu|_T = \text{id}$. We claim that μ is in $W(\Gamma)$. Take $x = 1$ and y a homogeneous element in \mathcal{C}_h of norm 1. Then the element $c = x \cdot y \in \text{Spin}(\mathcal{C}, n)$ gives rise to the automorphism ψ_c of \mathcal{A} — see step 3) in the proof of Theorem 4.2. The restriction $\psi_c|_{\iota_1(\mathcal{C})}$ is given by χ_c and hence stabilizes the homogeneous components in $\iota_1(\mathcal{C})$. For $z \in \mathcal{C}_u$ with $u \in T$, we have $\iota_2(z) \in \mathcal{A}_{b+u}$ and, using (17), we obtain $\psi_c(\iota_2(z)) = \iota_2(\rho_c^+(z)) = \iota_2(zy) \in \mathcal{A}_{b+u+h}$; also, $\iota_3(z) \in \mathcal{A}_{a+b+u}$ and hence we obtain $\psi_c(\iota_3(z)) = \iota_3(\rho_c^-(z)) = \iota_3(yz) \in \mathcal{A}_{a+b+u+h}$. Therefore, $\psi_c \in \text{Aut}(\Gamma)$, and its projection μ_c into $W(\Gamma)$ fixes a and the elements of T while taking b to $b + h$. By symmetry, we may also find an element in $W(\Gamma)$ which fixes b and the elements of T and takes a to $a + h$. Hence all matrices of the form $\begin{pmatrix} I & 0 \\ * & I \end{pmatrix}$ are contained in $W(\Gamma)$.

So far, we have proved that the stabilizer of T is contained in $W(\Gamma)$. But conversely, if $\psi \in \text{Aut}(\Gamma)$, then $\psi(\mathcal{A}_e) = \mathcal{A}_e$, so ψ permutes the idempotents E_i , $i = 1, 2, 3$, and hence induces a permutation of the elements $a, b, a + b$. By composing ψ with a suitable element of the stabilizer of T , we may assume that $\psi(E_i) = E_i$ for all $i = 1, 2, 3$. Then the projection of ψ into $W(\Gamma)$ preserves the cosets $a + T$, $b + T$ and $a + b + T$, and hence it preserves T . \square

Remark 4.5. As any $\psi \in \text{Stab}(\Gamma)$ fixes E_i and multiplies each $\iota_i(w_j)$, $i, j = 1, 2, 3$, by either 1 or -1 , we see that $\text{Stab}(\Gamma) = \text{Diag}(\Gamma)$ is isomorphic to \mathbb{Z}_2^5 .

4.4. $\mathbb{Z} \times \mathbb{Z}_2^3$ -grading

Theorem 4.6. Let Γ be the $\mathbb{Z} \times \mathbb{Z}_2^3$ -grading on the Albert algebra defined by (14) and (12) over an algebraically closed field of characteristic different from 2. Then $W(\Gamma) = \text{Aut}(\mathbb{Z} \times \mathbb{Z}_2^3)$.

Proof. Let T be the subgroup generated by c_j , $j = 1, 2, 3$. Then T is the torsion subgroup of $\mathbb{Z} \times \mathbb{Z}_2^3$ and hence we have $\mu(T) = T$ for all $\mu \in \text{Aut}(\mathbb{Z} \times \mathbb{Z}_2^3)$. Let $a = (1, \bar{0}, \bar{0}, \bar{0})$.

The group $\text{Aut}(\mathbb{Z} \times \mathbb{Z}_2^3)$ is generated by 1) the automorphism μ_0 that fixes T point-wise and takes a to $-a$, 2) the automorphisms β_h , for $h \in T$, that fix T point-wise and take a to $a + h$, and 3) the automorphisms of T extended to $\mathbb{Z} \times \mathbb{Z}_2^3$ by fixing a . We will show that all these automorphisms are contained in $\text{Aut}(\Gamma)$.

1) The order 2 automorphism ψ_0 of \mathcal{A} given by $S^\pm \mapsto S^\mp$, $v_\pm(x) \mapsto v_\mp(x)$, $v(a) \mapsto -v(a)$, for $x \in \mathbb{C}$ and $a \in \mathbb{C}_0$, belongs to $\text{Aut}(\Gamma)$, and its projection into $W(\Gamma)$ is precisely μ_0 .

2) Consider the T -grading Γ_0 on \mathbb{C} given by (12). Fix $h \in H$. Pick a norm 1 element $x \in \mathbb{C}_h$, then take $y \in \mathbb{C}_0$ homogeneous of norm 1 with $n(x, y) = 0$. Then $x = -xy^2 = -(xy)y = zy$, where $z = -xy$ is a homogeneous element of norm 1 in \mathbb{C}_0 . Note that $\deg y + \deg z = \deg x = h$. Consider the element $c = z \cdot y \in \text{Spin}(\mathbb{C}_0, n)$ and the associated automorphism ψ_c of \mathcal{A} . Then ψ_c stabilizes the homogeneous components in $v(\mathbb{C}_0)$, while $\psi_c(v_\pm(w)) = v_\pm(\rho_c^+(w)) = -v_\pm((wy)z)$ – see [EK12, Remark 6.4]. Hence ψ_c belongs to $\text{Aut}(\Gamma)$, and its projection into $W(\Gamma)$ fixes T point-wise and takes $a + u$ to $a + u + h$ for any $u \in T$. Thus, this projection is the desired element β_h .

3) Given any automorphism μ of T , Theorem 3.5 tells us that there is an automorphism φ of \mathbb{C} that belongs to $\text{Aut}(\Gamma_0)$ and whose projection into $W(\Gamma_0)$ is μ . The automorphism ψ of \mathcal{A} determined by $\psi(S^\pm) = S^\pm$, $\psi(v_\pm(x)) = v_\pm(\varphi(x))$, for all $x \in \mathbb{C}$, belongs to $\text{Aut}(\Gamma)$, and its projection into $W(\Gamma)$ is the automorphism of $\mathbb{Z} \times \mathbb{Z}_2^3$ fixing a and restricting to μ on T . \square

Remark 4.7. One can show that $\text{Stab}(\Gamma) = \text{Diag}(\Gamma)$, which is isomorphic to $\mathbb{F}^\times \times \mathbb{Z}_2^3$.

4.5. \mathbb{Z}_3^3 -grading

Recall that this grading occurs only if $\text{char } \mathbb{F} \neq 3$. Let $\Gamma: \mathcal{A} = \bigoplus_{g \in \mathbb{Z}_3^3} \mathcal{A}_g$ be the grading induced by the commuting order 3 automorphisms φ_i , $i = 1, 2, 3$, defined by (15) and (16), i.e.,

$$\varphi_i(X) = \omega^{k_i} X \quad \text{for all } X \in \mathcal{A}_{(\bar{k}_1, \bar{k}_2, \bar{k}_3)}, \quad (19)$$

where ω is a fixed primitive third root of unity. Let $\{g_1, g_2, g_3\}$ be a basis of \mathbb{Z}_3^3 and pick nonzero $X_i \in \mathcal{A}_{g_i}$, $i = 1, 2, 3$. Then X_1, X_2, X_3 generate the Albert algebra, and we can scale them so that $X_i^3 = 1$ and hence $N(X_i) = 1$, $i = 1, 2, 3$.

The subalgebra generated by X_3 is isomorphic to $\mathbb{F} \times \mathbb{F} \times \mathbb{F}$, so there exists an automorphism of \mathcal{A} sending it to $\mathbb{F}E_1 \oplus \mathbb{F}E_2 \oplus \mathbb{F}E_3$. Permuting E_1, E_2, E_3 if necessary, we may assume that $\varphi_3(E_i) = E_{i+1}$. In other words, we may assume $X_3 = \sum_{i=1}^3 \omega^{-i} E_i$. The subalgebra fixed by φ_3 is $\mathbb{F}1 \oplus \{\sum_{i=1}^3 \tilde{l}(x) \mid x \in \mathbb{C}\}$, so there are elements $x, y \in \mathbb{C}$ such that $X_1 = \frac{1}{2} \sum_{i=1}^3 \tilde{l}(x)$ and $X_2 = \frac{1}{2} \sum_{i=1}^3 \tilde{l}(y)$.

For any $z \in \mathbb{C}$, the norm of $\sum_{i=1}^3 \tilde{l}_i(z)$ is given by:

$$\begin{aligned} N(\tilde{l}_1(z) + \tilde{l}_2(z) + \tilde{l}_3(z)) &= N(\iota_1(\tau(z)) + \iota_2(\tau^2(z)) + \iota_3(z)) \\ &= 8n(\tau(z), \overline{\tau^2(z)\bar{z}}) = 8n(z, \overline{\tau(z)\tau^2(z)}) \\ &= 8n(z, z * z). \end{aligned}$$

Since $N(X_i) = 1$ for $i = 1, 2, 3$, we get $n(x, x * x) = 1 = n(y, y * y)$. And since $T(X_i^2) = 0$, we have $n(x) = 0 = n(y)$. Also, since the homogeneous components $\mathcal{A}_{(\pm\bar{1}, \bar{0}, \bar{0})}$ and $\mathcal{A}_{(\bar{0}, \pm\bar{1}, \bar{0})}$ are orthogonal relative to the trace form T , we conclude that the subspaces $\mathbb{F}x \oplus \mathbb{F}x * x$ and $\mathbb{F}y \oplus \mathbb{F}y * y$ are orthogonal relative to the norm of \mathbb{C} . Now [Eld09, Proposition 3.9 and Theorem 3.12] show that either $x * y = 0$ or $y * x = 0$, but not both, and that x, y generate the Okubo algebra $(\mathbb{C}, *, n)$ with multiplication table independent of x and y .

Fix $a, b \in \mathbb{C}$ with $a * b = 0$, $n(a) = 0 = n(b)$, $n(a, a * a) = 1 = n(b, b * b)$, and $n(\mathbb{F}a + \mathbb{F}a * a, \mathbb{F}b + \mathbb{F}b * b) = 0$. Let Γ^+ and Γ^- be two \mathbb{Z}_3^3 -gradings on the Albert algebra that are determined by the following conditions:

$$\begin{aligned}
\Gamma^+: \quad & \deg\left(\sum_{i=1}^3 \tilde{t}_i(a)\right) = (\bar{1}, \bar{0}, \bar{0}), & \Gamma^-: \quad & \deg\left(\sum_{i=1}^3 \tilde{t}_i(a)\right) = (\bar{0}, \bar{1}, \bar{0}), \\
& \deg\left(\sum_{i=1}^3 \tilde{t}_i(b)\right) = (\bar{0}, \bar{1}, \bar{0}), & & \deg\left(\sum_{i=1}^3 \tilde{t}_i(b)\right) = (\bar{1}, \bar{0}, \bar{0}), \\
& \deg\left(\sum_{i=1}^3 \omega^{-i} E_i\right) = (\bar{0}, \bar{0}, \bar{1}), & & \deg\left(\sum_{i=1}^3 \omega^{-i} E_i\right) = (\bar{0}, \bar{0}, \bar{1}).
\end{aligned} \tag{20}$$

For example, we may take $a = e_1$ and $b = u_1$. Then $\Gamma^+ = \Gamma$.

Note that for Γ^+ with $\{g_1, g_2, g_3\}$ being the standard basis of \mathbb{Z}_3^3 , we have:

$$\begin{aligned}
(X_1 X_2) X_3 &= \left(\frac{1}{4} \sum_{i=1}^3 \tilde{t}_i(b * a) \right) X_3 \\
&= \frac{1}{8} \sum_{i=1}^3 (\omega^{-(i+1)} + \omega^{-(i+2)}) \tilde{t}_i(b * a) \\
&= -\frac{1}{8} \sum_{i=1}^3 \omega^{-i} \tilde{t}_i(b * a),
\end{aligned}$$

while

$$\begin{aligned}
X_1(X_2 X_3) &= X_1 \left(\frac{1}{4} \sum_{i=1}^3 (\omega^{-(i+1)} + \omega^{-(i+2)}) \tilde{t}_i(y) \right) \\
&= X_1 \left(-\frac{1}{4} \sum_{i=1}^3 \omega^{-i} \tilde{t}_i(y) \right) \\
&= -\frac{1}{8} (\omega^{-2} \tilde{t}_1(b * a) + \omega^{-3} \tilde{t}_2(b * a) + \omega^{-1} \tilde{t}_3(b * a)),
\end{aligned}$$

so that $(X_1 X_2) X_3 = \omega X_1(X_2 X_3)$. However, for Γ^- with $\{g_1, g_2, g_3\}$ being the standard basis of \mathbb{Z}_3^3 , analogous computations give $(X_1 X_2) X_3 = \omega^{-1} X_1(X_2 X_3)$. Therefore Γ^+ and Γ^- are not isomorphic.

Theorem 4.8. *Let Γ be the \mathbb{Z}_3^3 -grading on the Albert algebra as in (19) over an algebraically closed field of characteristic different from 2 and 3. Then $W(\Gamma)$ is the commutator subgroup of $\text{Aut}(\mathbb{Z}_3^3)$, i.e., $W(\Gamma) \cong \text{SL}_3(3)$.*

Proof. We may assume $\Gamma = \Gamma^+$ as in (20). Identify $\text{Aut}(\mathbb{Z}_3^3)$ with $\text{GL}_3(3)$. For any $\mu \in \text{GL}_3(3)$, let g_j , $j = 1, 2, 3$, be the images of the elements of the standard basis (i.e., the columns of matrix μ). Pick elements X'_j such that $\deg X'_j = g_j$ and $(X'_j)^3 = 1$, $j = 1, 2, 3$. We have shown that there exists an automorphism of \mathcal{A} either sending X'_j to X_j associated with Γ^+ , $j = 1, 2, 3$, or sending X'_j to X_j associated with Γ^- , $j = 1, 2, 3$, but not both. This shows that $\text{Aut}(\Gamma)$ has index 2 in $\text{GL}_3(3)$. Since the commutator subgroup of $\text{GL}_3(3)$ is $\text{SL}_3(3)$, and it has index 2, we conclude that $W(\Gamma) = \text{SL}_3(3)$. \square

Remark 4.9. Clearly, $\text{Stab}(\Gamma) = \text{Diag}(\Gamma)$ is isomorphic to \mathbb{Z}_3^3 .

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